Chapter 7

Experimental Designs and Analysis of Variance

Terminology

Experiment - a well-defined process or planned inquiry undertaken to obtain new facts or to confirm or deny the results of previous studies.

Factor - an independent variable whose effect on the dependent variable/s we want to study.

Treatment - level of a factor or combination of factors whose effect we want to measure and compare with others.

Experimental unit - material to which a given treatment is applied; an object, element, or any entity which receives a treatment and will manifest its effects.

Replication - number of times a treatment is applied to experimental units in an experiment.

Experimental design - includes the plan and actual procedure of conducting the experiment.

Experimental layout - refers to the placement of treatments on the experimental units.

Analysis of variance (ANOVA) - a method for dividing the variation observed from experimental data into different parts, each part assignable to a known source, cause, or factor.

Example 1:
Three dosages (5mg, 10mg, 15mg) of a cream is being evaluated for its effect against a particular skin problem. Fifteen randomly selected women (age range: 20-29) who have the skin problem participated in the study. Each dosage was randomly assigned to five women and an “effect score” was recorded for each after the duration of the experiment.

Factor: Dosages of the cream

Treatments: T₁=5mg, T₂=10mg, T₃=15mg

Replications: 5 women per dosage/treatment

Experimental unit: a woman

Response variable: effect score
Terminology

Example 2:
An experiment in agronomy is conducted to evaluate the effect of complete fertilizer and irrigation on the yield of rice. Three levels of complete fertilizer (50kg/ha, 100kg/ha, 150kg/ha) and two levels of irrigation (5 cm and 10 cm) were used in the experiment. Each combination of fertilizer and irrigation is applied to three plots. At the end of the experiment the yield of rice is recorded.

Completely Randomized Design

Advantages of the design
1. Simplest in design and in data analysis
2. Allows for maximum degrees of freedom in the error term
3. The treatments may or may not be equally replicated

Disadvantages of the design
1. Requires a large amount of homogeneous experimental units
   - if there are many treatments it is not advisable to use CRD
2. Suitable only in laboratory and greenhouse experiments where the environment and other conditions can be controlled

<table>
<thead>
<tr>
<th>Treatment</th>
<th>Replication</th>
<th>Treatment Total</th>
<th>Treatment Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>T_1</td>
<td>23 36 31 33 4</td>
<td>123</td>
<td>30.75</td>
</tr>
<tr>
<td>T_2</td>
<td>42 26 47 34 4</td>
<td>149</td>
<td>37.25</td>
</tr>
<tr>
<td>T_3</td>
<td>47 43 43 39 4</td>
<td>172</td>
<td>43</td>
</tr>
<tr>
<td>TOTAL</td>
<td></td>
<td>444</td>
<td>37</td>
</tr>
</tbody>
</table>
**Completely Randomized Design**

### Treatment Replication

<table>
<thead>
<tr>
<th>Treatment</th>
<th>Replication</th>
<th>Treatment Total</th>
<th>Treatment Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>T₁</td>
<td>2.0</td>
<td>2.2</td>
<td>1.8</td>
</tr>
<tr>
<td>T₂</td>
<td>1.7</td>
<td>1.9</td>
<td>1.5</td>
</tr>
<tr>
<td>T₃</td>
<td>2.0</td>
<td>2.4</td>
<td>2.7</td>
</tr>
<tr>
<td>T₄</td>
<td>2.1</td>
<td>2.2</td>
<td>2.2</td>
</tr>
<tr>
<td>TOTAL</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Completely Randomized Design**

### Linear Model:

\[ y_{ij} = \mu + \tau_i + \epsilon_{ij} \]

where:

- \( y_{ij} \) = the response of the \( j^{th} \) experimental unit given treatment \( i \), \( i = 1, 2, \ldots, t \) & \( j = 1, 2, \ldots, r_i \)
- \( \mu \) = overall mean
- \( \tau_i \) = effect of \( i^{th} \) treatment
- \( \epsilon_{ij} \) = experimental error

\[ \epsilon_{ij} \sim N(0, \sigma^2) \]

**Completely Randomized Design**

### Assumptions

1. The \( t \) treatments represent random samples taken independently from \( t \) different populations
2. Each of the \( t \) populations are normal
3. The \( t \) populations have the same variance
4. Treatment effect and error effect are additive

\[ y_{ij} \sim NID(\mu, \sigma^2) \]

**Completely Randomized Design**

### Objectives of the analysis

1. To test the hypothesis of equality of treatment means
   
   \( H_0: \mu_1 = \mu_2 = \ldots = \mu_t \)
   
   \( H_a: \) At least two treatment means are not equal.
   
   or,
   
   \( H_0: \tau_i = 0 \) for all \( i \)
   
   \( H_a: \tau_i \neq 0 \) for at least one \( i \)

2. To estimate the magnitude of the effect of the \( i^{th} \) treatment on the dependent variable \( Y \)
Types of Effects Models

1. Fixed Effects Model
   - the treatments actually used in the experiment are chosen by the researcher
   - conclusions will apply only to the treatments used in the experiment

2. Random Effects Model
   - the treatments used in the experiment constitute a random sample of treatments from a large population of treatments
   - conclusions applies to all treatments in the population of treatments

Analysis of Variance (ANOVA)

- a method of partitioning the total variation observed in the response variable into components which can be attributed to a known source of variation

Components of Total Variation

1. Between treatment variability
   - Variation of each treatment mean from the overall mean
   - Treatment effect

2. Within treatment variability
   - Variation of each observation from the mean of the treatment group where they belong
   - Experimental error

Notations

\[ y_{ij} = \text{response of the } j^{th} \text{ experimental unit applied with treatment } i \]
\[ y_i = \text{total response of all units applied with treatment } i \]
\[ y = \text{total response of all units across all treatments} \]
\[ y_{..} = \text{mean response of all units applied with treatment } i \]
\[ y_{..} = \text{mean response of all units across all treatments} \]
\[ r_i = \text{number of replications or experimental units applied with treatment } i \]
\[ t = \text{number of treatments} \]
\[ n = \text{total number of observations/ experimental units} \]
Completely Randomized Design

Computational Formulas for the Sums of Squares

\[ CF = \frac{\sum_{i}^{t} \sum_{j}^{n} y_{ij}^2}{n}, \text{ correction factor} \]

\[ TSS = \sum_{i=1}^{t} \sum_{j=1}^{r_{i}} y_{ij}^2 - CF, \text{ total sum of squares} \]

\[ SSTr = \sum_{i=1}^{t} \frac{y_{i}^2}{r_{i}} - CF, \text{ treatment sum of squares} \]

\[ SSE = TSS - SSTr, \text{ experimental error sum of squares} \]

ANOVA Table

<table>
<thead>
<tr>
<th>Source of variation</th>
<th>df</th>
<th>Sum of Squares</th>
<th>Mean Squares</th>
<th>( F_{c} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatment</td>
<td>t-1</td>
<td>SSTr</td>
<td>MSTr</td>
<td>MSTr/MSE</td>
</tr>
<tr>
<td>Experimental Error</td>
<td>n-t</td>
<td>SSE</td>
<td>MSE</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>n-1</td>
<td>TSS</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ MSTr = \frac{SSTr}{t-1} \]

\[ MSE = \frac{SSE}{n-t} \]

Structure of the F-Ratio (F Statistic)

\[ F = \frac{\text{variance between treatments}}{\text{variance within treatments}} = \frac{MSTr}{MSE} \]

- If F close to 1 then treatment effect is zero or negligible
- If F > 1 then treatment effect is large

Test of significance of the difference in the treatment means

\[ \text{Ho: } \mu_1 = \mu_2 = \ldots = \mu_t \]
\[ \text{Ha: At least two treatment means are not equal.} \]

Test of significance of treatment effects

\[ \text{Ho: } \tau_1 = \tau_2 = \ldots = \tau_t = 0 \]
\[ \text{Ha: At least one } \tau_i \text{ is different from zero} \]

Rejection Rule

Reject Ho if \( F_c \geq F_{\alpha,(t-1,n-t)} \); otherwise, do not reject Ho.
Coefficient of variation (CV)

- a measure of the degree of the precision of the experiment
- an overall measure of the reliability of the experiment

\[ CV = \frac{\sqrt{MSE}}{y} \times 100\% \]

A CV of at most 10% indicates that the experiment is precise. This means a very small experimental error.

**Example:**

Consider a CRD experiment conducted to determine the effect of temperature (°C) on the dissolved amount of a chemical compound in 100 g of water. Each of the temperatures used in the experiment (0, 15, 30, 45, 60, 70°C) was replicated three times. The experimental layout may be the following.

<table>
<thead>
<tr>
<th>Temperature (°C)</th>
<th>Replication</th>
<th>Treatment Total</th>
<th>Treatment Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>8</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>15</td>
<td>12</td>
<td>10</td>
<td>14</td>
</tr>
<tr>
<td>30</td>
<td>25</td>
<td>21</td>
<td>24</td>
</tr>
<tr>
<td>45</td>
<td>31</td>
<td>33</td>
<td>28</td>
</tr>
<tr>
<td>60</td>
<td>44</td>
<td>39</td>
<td>42</td>
</tr>
<tr>
<td>75</td>
<td>48</td>
<td>51</td>
<td>44</td>
</tr>
</tbody>
</table>

Total: \( n = 18 \)  
\( \bar{y} = 488 \)  
\( \bar{y} \approx 27.11 \)

**Example:**

\[ CF = \frac{y^2}{n} = \frac{488^2}{18} \approx 13230.22 \]

\[ TSS = \sum_{i=1}^{t} \sum_{j=1}^{r_i} y_{ij}^2 - CF = \left( 8^2 + 12^2 + \ldots + 44^2 \right) - CF \]
\[ = 17142 - 13230.22 = 3911.78 \]

\[ SSTr = \sum_{i=1}^{t} \sum_{j=1}^{r_i} \frac{y_{ij}^2}{r_i} - CF = \left( \frac{22^2}{3} + \frac{36^2}{3} + \ldots + \frac{143^2}{3} \right) - CF \]
\[ = 17072.67 - 13230.22 = 3845.45 \]

\[ SSE = TSS - SSTr = 3911.78 - 3845.45 = 69.33 \]
### Completely Randomized Design

#### ANOVA Table

<table>
<thead>
<tr>
<th>Source of variation</th>
<th>df</th>
<th>Sum of Squares</th>
<th>Mean Squares</th>
<th>$F_c$</th>
<th>$F_{0.05}$</th>
<th>$F_{0.01}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temperatures</td>
<td>5</td>
<td>3842.45</td>
<td>768.49</td>
<td>132.96**</td>
<td>3.11</td>
<td>5.06</td>
</tr>
<tr>
<td>Experimental Error</td>
<td>12</td>
<td>69.33</td>
<td>5.78</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>17</td>
<td>3911.78</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

ns: not significant
* significant at 5% level of significance
** significant at 1% level of significance

### Completely Randomized Design

$H_0$: There is no difference in the mean dissolved amount of the chemical among the six temperature levels

$(\mu_1 = \mu_2 = \ldots = \mu_6)$

$H_a$: There is difference in the mean dissolved amount of the chemical among the six temperature levels

Equivalently,

$H_0$: Temperature has no significant effect on the dissolved amount of the chemical. $(\tau_1 = \tau_2 = \ldots = \tau_6 = 0)$

$H_a$: Temperature has significant effect on the dissolved amount of the chemical.

### Randomized Complete Block Design

- when we recognize one extraneous source of variation which causes the heterogeneity of the experimental units
- blocking or stratification is deliberately done to classify the experimental units into homogeneous groups or blocks.
- the number of experimental units in each block is equal to the number of treatments since all the treatments should be accommodated in each block
- the treatments are randomly allotted to the experimental units within each block
- each block is a replicate so that the number of replications per treatment is equal to the number of blocks
Randomized Complete Block Design

Advantages of the design
1. The design is easy to conduct.
2. The statistical analysis is simple.
3. It can be used to accommodate any number of treatments in any number of blocks.

Disadvantage
Since the experimental units within blocks must be homogeneous, the design is suited for a relatively small number of treatments.

Randomized Complete Block Design

Linear Model:

\[ y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij} \]

where:
- \( y_{ij} \) = response of the experimental unit in the \( j^{th} \) block applied with treatment \( i \)
- \( \mu \) = overall mean
- \( \tau_i \) = effect of the \( i^{th} \) treatment, \( i = 1, 2, \ldots, t \)
- \( \beta_j \) = effect of the \( j^{th} \) block, \( j = 1, 2, \ldots, b \)
- \( \epsilon_{ij} \) = random error term
- \( \epsilon_{ij} \sim NID(0, \sigma^2) \)

\[ \sum_i \tau_i = \sum_j \beta_j = 0 \]

Randomized Complete Block Design

Notations
- \( y_{ij} \) = response of the experimental unit in the \( j^{th} \) block applied with treatment \( i \)
- \( y_i \) = total response of all units applied with treatment \( i \)
- \( y_j \) = total response of all units in \( j^{th} \) block
- \( y \) = total response of all units in the experiment
- \( \bar{y}_{ij} \) = mean response of all units applied with treatment \( i \)
- \( \bar{y}_j \) = mean response of all units in \( j^{th} \) block
- \( \bar{y} \) = mean response of all units in the experiment
- \( b \) = number of blocks
- \( t \) = number of treatments
- \( n = tb \) = total number of observations/expperimental units

Randomized Complete Block Design

Assumptions
1. Each observation \((y_{ij})\) is randomly taken from the \( tb^{th}\) population; that is, there are \( tb \) populations and one observation \((y_{ij})\) is obtained randomly from each of the \( tb \) populations.
2. Each of the \( tb \) populations is normal.
3. The \( tb \) populations have equal variance, \( \sigma^2 \)
4. Treatment effect, block effect, and error effect are additive.

\[ y_{ij} \sim NID(\mu_{ij}, \sigma^2) \]
Objectives of the analysis

1. To test the hypothesis of equality of treatment means
   \[ H_0: \mu_1 = \mu_2 = \ldots = \mu_p \]
   \[ H_a: \text{At least two treatment means are not equal.} \]
   or,
   \[ H_0: \tau_i = 0 \text{ for all } i \]
   \[ H_a: \tau_i \neq 0 \text{ for at least one } i \]

2. To estimate the magnitude of the effect of the \( i \)th treatment on the dependent variable \( Y \)
   \[ \text{Treatment effect (} \tau_i \text{): } \hat{\tau}_i = \bar{Y}_i - \bar{Y}.. \]
   \[ \text{Block effect (} \beta_j \text{): } \hat{\beta}_j = \bar{Y}_j - \bar{Y}.. \]

Randomized Complete Block Design

Computational Formulas for the Sums of Squares

\[ CF = \frac{\sum_{i=1}^{t} \sum_{j=1}^{b} y_{ij}^2}{n}, \text{ correction factor} \]

\[ TSS = \sum_{i=1}^{t} \sum_{j=1}^{b} y_{ij}^2 - CF, \text{ total sum of squares} \]

\[ SSTr = \frac{1}{b} \sum_{i=1}^{t} \sum_{j=1}^{b} y_{ij}^2 - CF, \text{ treatment sum of squares} \]

\[ SSB = \frac{1}{t} \sum_{i=1}^{b} y_{ij}^2 - CF, \text{ block sum of squares} \]

\[ SSE = TSS - SSTr - SSB, \text{ expil. error sum of squares} \]

ANOVA Table

<table>
<thead>
<tr>
<th>Source of variation</th>
<th>df</th>
<th>Sum of Squares</th>
<th>Mean Squares</th>
<th>( F_c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatments</td>
<td>( t-1 )</td>
<td>SSTr</td>
<td>MSTr</td>
<td>MSTr/MSE</td>
</tr>
<tr>
<td>Blocks</td>
<td>( b-1 )</td>
<td>SSB</td>
<td>MSB</td>
<td></td>
</tr>
<tr>
<td>Error (Residual)</td>
<td>((b-1)(t-1))</td>
<td>SSE</td>
<td>MSE</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>( tb-1 )</td>
<td>TSS</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note:
Sometimes, we may want to determine if blocking is effective in reducing experimental error. This may be done using an F test, where \( F_c = \frac{\text{MSB}}{\text{MSE}} \).
Randomized Complete Block Design

<table>
<thead>
<tr>
<th>Fertilizer Level</th>
<th>Variety of Wheat</th>
<th>Treatment Total</th>
<th>Treatment Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>V₁</td>
<td>V₂</td>
<td>V₃</td>
</tr>
<tr>
<td>T₁</td>
<td>64</td>
<td>72</td>
<td>74</td>
</tr>
<tr>
<td>T₂</td>
<td>55</td>
<td>57</td>
<td>47</td>
</tr>
<tr>
<td>T₃</td>
<td>59</td>
<td>66</td>
<td>58</td>
</tr>
<tr>
<td>T₄</td>
<td>58</td>
<td>57</td>
<td>53</td>
</tr>
<tr>
<td>Block Total</td>
<td>236</td>
<td>252</td>
<td>232</td>
</tr>
<tr>
<td>Block Mean</td>
<td>59</td>
<td>63</td>
<td>58</td>
</tr>
</tbody>
</table>

\[ \bar{y}_{..} = 720 \quad \bar{y}_{..} = 60 \]

Randomized Complete Block Design

**ANOVA Table**

<table>
<thead>
<tr>
<th>Source of variation</th>
<th>df</th>
<th>Sum of Squares</th>
<th>Mean Squares</th>
<th>( F_c )</th>
<th>( F_{0.05} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fertilizer Levels</td>
<td>3</td>
<td>498</td>
<td>166</td>
<td>9.22*</td>
<td>4.76</td>
</tr>
<tr>
<td>Varieties</td>
<td>2</td>
<td>56</td>
<td>28</td>
<td>1.56ns</td>
<td>5.14</td>
</tr>
<tr>
<td>Error (Residual)</td>
<td>6</td>
<td>108</td>
<td>18</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>11</td>
<td>662</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Randomized Complete Block Design

Test of hypothesis of equality of treatment means

Ho: \( \mu_1 = \mu_2 = \mu_3 = \mu_4 \)

Ha: At least two treatment means are not equal.

or,

Ho: \( \tau_i = 0 \) for all \( i \)

Ha: \( \tau_i \neq 0 \) for at least one \( i \)

\( \alpha = 0.05 \)

Test stat.: \( F \)

\[ F_c = \frac{\text{MSTr}}{\text{MSE}} = 9.22 \]

Decision Rule: Reject Ho if \( F_c \geq F_{0.05}(3,6) = 4.76 \); otherwise, do not reject Ho.

Decision: Since 9.22 > 4.76 reject Ho.

Conclusion: The mean yield of wheat is not the same for all the four fertilizer levels at the 5% level of significance.

\[ CF = \frac{y^2}{pb} = \frac{(720)^2}{(4)(3)} = 43200 \]

\[ TSS = \sum_{i=1}^{p} \sum_{j=1}^{b} y_{ij}^2 - CF = 64^2 + 72^2 + ... + 53^2 - CF \]

= 43862 - 43200 = 662

\[ \text{SSTr} = \frac{1}{b} \sum_{i=1}^{p} \sum_{j=1}^{b} y_{ij}^2 - C.F. = \frac{1}{3} \left( 210^2 + 159^2 + 183^2 + 168^2 \right) - CF \]

= \frac{1}{3} (131094) - 43200 = 498

\[ \text{SSB} = \frac{1}{p} \sum_{i=1}^{p} \sum_{j=1}^{b} y_{ij}^2 - C.F. = \frac{1}{4} \left( 236^2 + 252^2 + 232^2 \right) - CF \]

= \frac{1}{4} (173024) - 43200 = 56

\[ \text{SSE} = \text{TSS} - \text{SSTr} - \text{SSB} = 662 - 498 - 56 = 108 \]
Testing if blocking is effective

Ho: \( \beta_j = 0 \) for all \( j \)
Ha: \( \beta_j \neq 0 \) for at least one \( j \)

\( \alpha = 0.05 \)

Test stat.: \( F \)

\[ F_c = \frac{\text{MSB}}{\text{MSE}} = 1.56 \]

Decision Rule: Reject Ho if \( F \geq F_{0.05, (2, 6)} = 5.14 \); otherwise, do not reject Ho.

Decision: Since \( 1.56 < 5.14 \) do not reject Ho.

Conclusion: There is no significant difference in the mean yield of wheat among the three varieties at the 5% level of significance. \textit{This means that there is no block effect or blocking is not effective.}